Tractability in Structured Probability Spaces

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PSDD Overview

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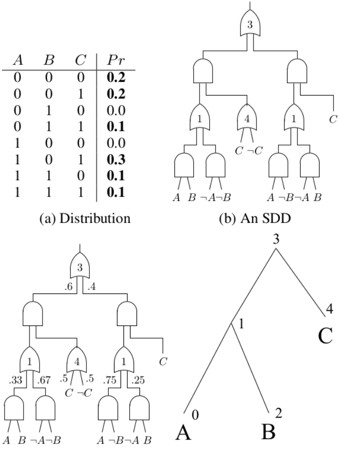

A	B	C	Pr
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0	1	1	**0.1**
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1	0	1	**0.3**
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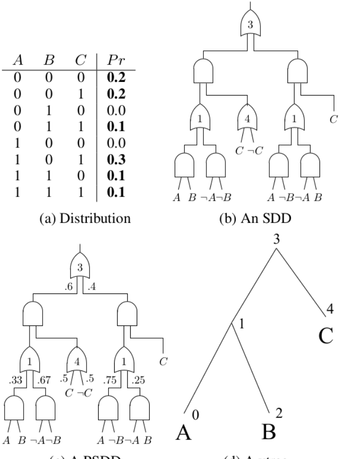

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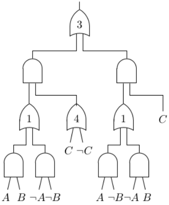

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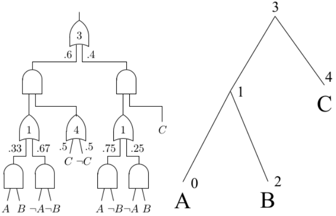

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- The probability of a complete instantiation x: Perform a bottom-up pass. Value of AND gate is product of its inputs. Value of OR gate is weighted sum of its inputs. Also, $\sum_{\mathbf{x}} Pr(\mathbf{x}) = 1$

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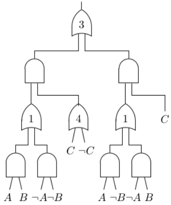

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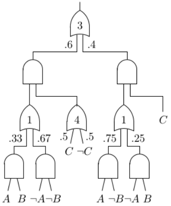

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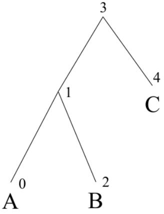

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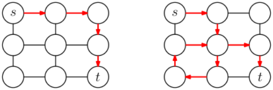
- 

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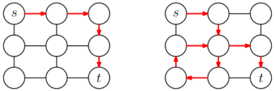
- 

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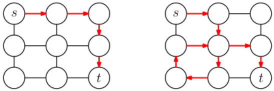
- 

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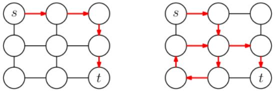
- 

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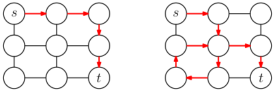
- 

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- If more than 2 variables of $\mathbf{B_i}$ are true in some x, then $Pr(x) = 0$.

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- When simple routes are also simple in G_B? Since $Pr(\gamma_G) + Pr(\kappa_G) = 1$, then if $Pr(\gamma_G) \approx 1$, then we expect the hierarchical distribution to be accurate.

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  - Multiply all the component PSDDs to get a single PSDD over the structured space of hierarchical simple-routes.