SPUDD: Stochastic Planning using Decision Diagrams

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Paper by: Hoey et al, UAI 1999

30th Nov, 2020

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- Derives from SPI which uses decision trees (unscalable) to represent π and V.
- Disjunctive structure in probability exploited by decision graphs.

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- Bellman Equation: $V_\pi(s) = R(s) + \gamma \sum_{t \in \mathcal{S}} Pr(s, \pi(s), t).V_\pi(t)$

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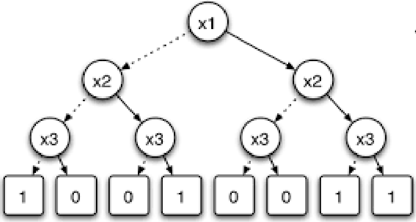
- Stopping criterion: $||V^{n+1} - V^n|| < \frac{\epsilon(1-\gamma)}{2\gamma}$
  $||X|| = \max\{|x| : x \in X\}$. Resulting π is ϵ-opt and V^{n+1} is within $\epsilon/2$ of V^*

ADD

- BDD: A function, $f : \mathcal{B}^n \rightarrow \mathcal{B}$
- ADD: Generalize BDD, $f : \mathcal{B}^n \rightarrow \mathcal{R}$.
    - Terminal node: $f(.) = c$
    - Non-terminal node: $f(x_1 \ldots x_n) = x_1.f_{then}(x_2 \ldots x_n) + \overline{x_1}.f_{else}(x_2 \ldots x_n)$



x1	x2	x3	f
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Figure 1: Binary Decision Diagram

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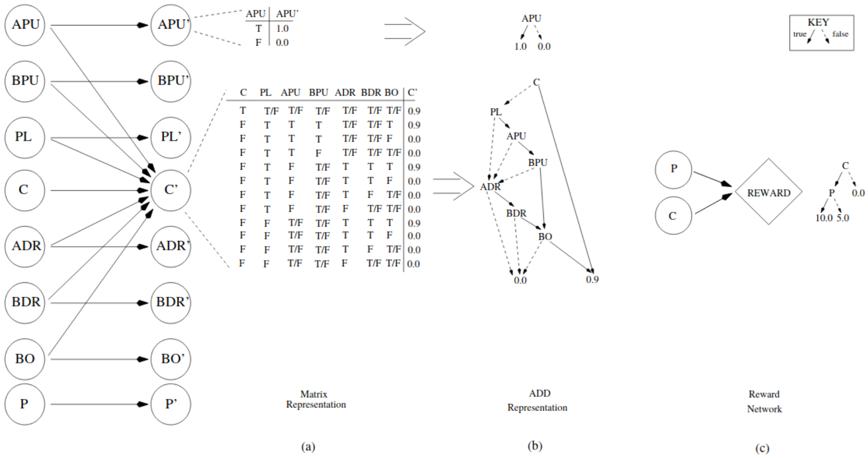
- Directed arcs from variables in X to variables in X′ denote direct causal relationship.
- CPT for each post-action variable X'_i defines a conditional distribution $P^a_{X'_i}$ over X'_i: $P^a_{X'_i}(X_1 \dots X_n)$.

Example

- Process planning problem: A factory agent is tasked to connect 2 objects A and B.
- One way the agent can connect is take take action *bolt*.
- State C (objects connected) is independent of variable P (objects painted).
- If obj A is punched (APU) after bolting depends only on whether it was punched before bolting.
- Use ADDs to represent the functions $P^a_{X'_i}$ (to capture regularities in the CPTs)
- ADDs also exploit context-specific independence in the distributions.

Example



C	PL	APU	BPU	ADR	BDR	BO	C'
T	T/F	T/F	T/F	T/F	T/F	T/F	0.9
F	T	T	T	T/F	T/F	T	0.9
F	T	T	T	T/F	T/F	F	0.0
F	T	T	F	T/F	T/F	T/F	0.0
F	T	F	T/F	T	T	T	0.9
F	T	F	T/F	T	T	F	0.0
F	T	F	T/F	T	F	T/F	0.0
F	T	F	T/F	F	T/F	T/F	0.0
F	F	T/F	T/F	T	T	T	0.9
F	F	T/F	T/F	T	T	F	0.0
F	F	T/F	T/F	T	F	T/F	0.0
F	F	T/F	T/F	F	T/F	T/F	0.0

Figure 2: Small FACTORY example: (a) action network for action *bolt*; (b) ADD representation of CPTs (action diagrams); and (c) immediate reward network and ADD representation of the reward table.

Example

- Regularity in CPT: $Pr_{C'}^{bolt}(C, PL, APU, BPU, ADR, BDR, BO) = [C + \overline{C}[(PL \cdot \overline{APU} \cdot \overline{PL}) \cdot ADR \cdot BDR + PL \cdot APU \cdot BPU] \cdot BO] \cdot 0.9$
- Reward function as ADD: $R(C, P) = C \cdot P \cdot 10 + C \cdot \overline{P} \cdot 5$
- Disjunctive structure exploited by ADD. Eg., CPT for C': Variety of distinct conditions each give give rise to successfully connecting the 2 parts. (Similar to paths)
- ADDs more compact than trees (and tables): 7 internal nodes and 2 leaves vs 11 internal nodes and 12 leaves. Std matrix: 128 parameters.
- ADDs more compact than trees *most* times but not always.

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- Savings both in space and computational time.
- V at each step is represented as an ADD. ($V^0 = R$)
- Exploit ADD structure of V^i and MDP representation to get ADD structure for V^{i+1}.

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  - Dual action diagrams: $Q^a_{X_i'}(X_i'; X_1 \dots X_n) = X_i' \cdot P^a_{X_i'}(X_1 \dots X_n) + \overline{X_i'} \cdot \overline{P^a_{X_i'}}(X_1 \dots X_n)$

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- Intuitively, Q denotes $P(X_i' = x_i' | X_1 = x_1 \ldots X_n = x_n)$ (under action a)
- To generate $V_a^{i+1}(s)$: Combine $V_a^i(t)$ with probability of reaching t from s.

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- To get V_a^{i+1}: Multiply dual action diagrams X'_j by V'^i and then eliminate X'_j.
- $Q^a_{X'_j} \cdot V'^i = f(X'_1 \dots X'_n, X_1 \dots X_n)$
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- Elimination of X'_j (Summing over left and right subgraphs of the ADD for f)
- $g(X'_1 \ldots X'_{j-1}, X'_{j+1]} \ldots X'_n, X_1 \ldots X_n) = \sum_{x'_j} V'^i(X'_1 \ldots x'_j \ldots X'_n) P(x'_j | X_1 \ldots X_n)$

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- Extensions to other dynamic programming algorithms.