Path Integral Formulation of Quantum Mechanics

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Outline

Introduction

Propagator

Evolution Operator

The matrix element

Moving forward (deriving the path integral)

Math trickery

Final touch

Let us go back to the year 1948...

REVIEWS OF
MODERN PHYSICS

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Space-Time Approach to Non-Relativistic Quantum Mechanics

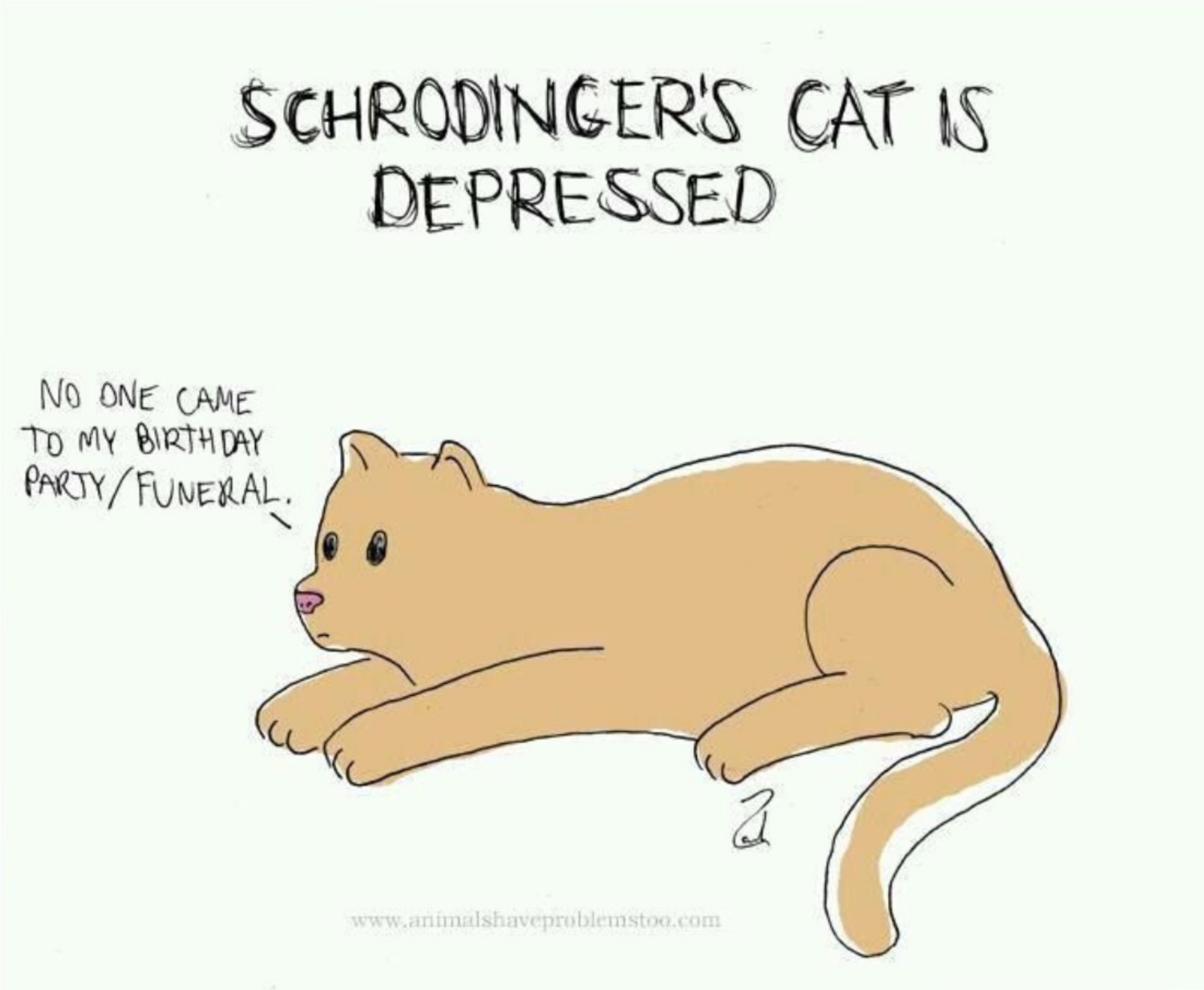
R. P. FEYNMAN

Cornell University, Ithaca, New York

Non-relativistic quantum mechanics is formulated here in a different way. It is, however, mathematically equivalent to the familiar formulation. In quantum mechanics the probability of an event which can happen in several different ways is the absolute square of a sum of complex contributions, one from each alternative way. The probability that a particle will be found to have a path $x(t)$ lying somewhere within a region of space time is the square of a sum of contributions, one from each path in the region. The contribution from a single path is postulated to be an exponential whose (imaginary) phase is the classical action (in units of $\hbar$) for the path in question. The total contribution from all paths reaching x, t from the past is the wave function $\psi(x, t)$. This is shown to satisfy Schroedinger's equation. The relation to matrix and operator algebra is discussed. Applications are indicated, in particular to eliminate the coordinates of the field oscillators from the equations of quantum electrodynamics.

Introduction: QM theories

Schroedinger's differential equation

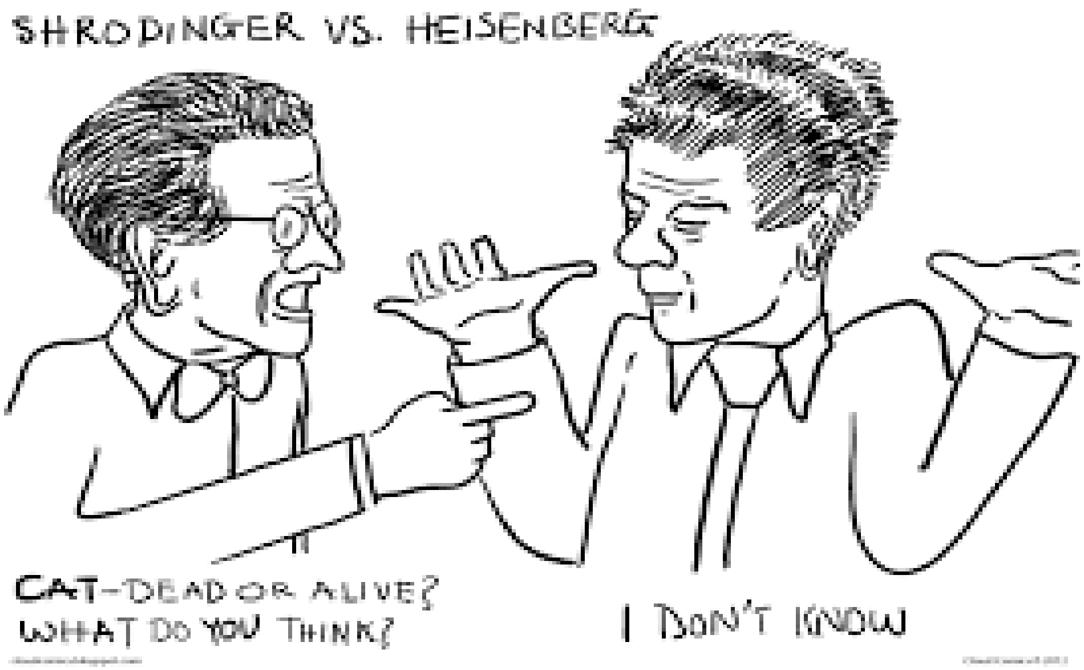


Introduction: QM theories

Heisenberg's matrix algebra

Introduction: QM theories

Feynman's path integral

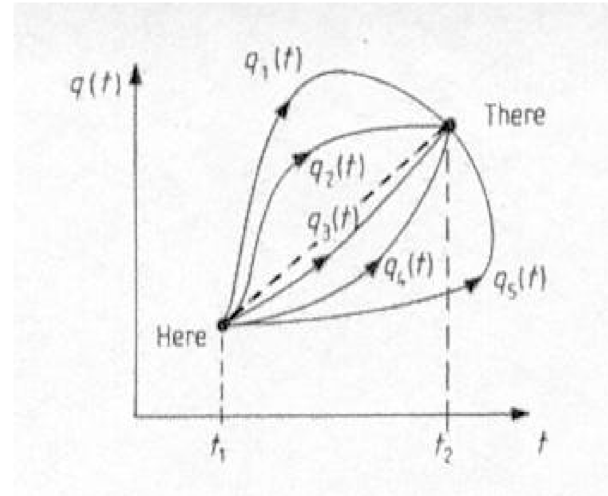


Brief overview: New formulation

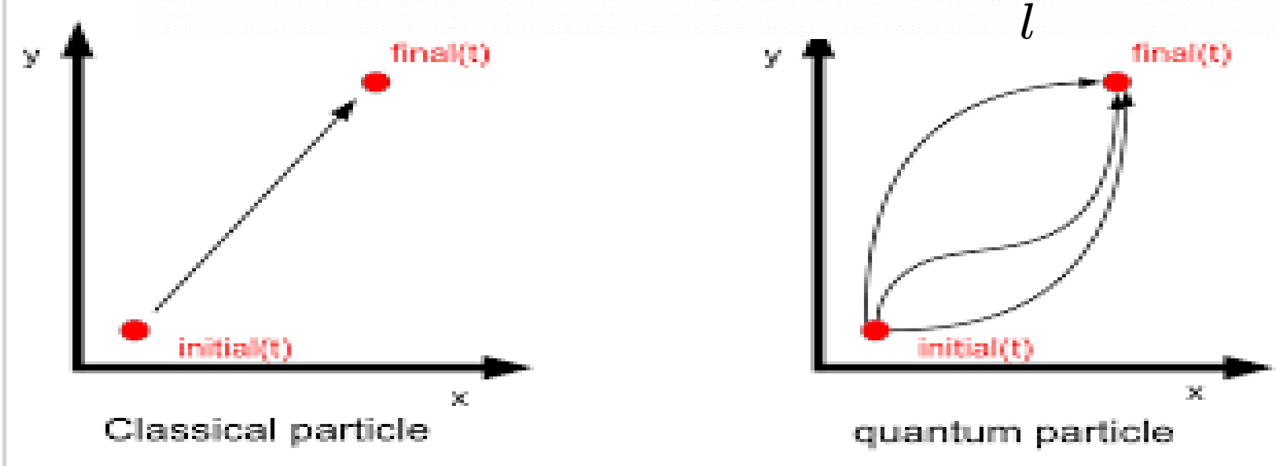
Aim: Transition amplitude of a quantum particle from an initial point R_i to a final point R_f

Notion of a trajectory: Particle goes through each of the possible paths at the same time.

Weights associated with each path: $e^{\frac{i}{\hbar}S}$



Probability: $$Pr(R_i \rightarrow R_f) = w_{i \rightarrow f} = \left| \sum_l e^{\frac{i}{\hbar}S_l} \right|^2$$

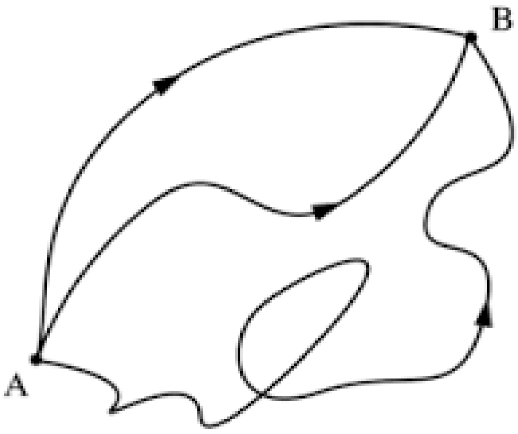


Propagator

Consider a particle localized at a point R_i at t=0 and let it evolve according to the Schroedinger equation.

$$i\hbar\partial_t|\Psi\rangle = \left[\frac{\hat{p}^2}{2m} + \hat{V}(R)\right]|\Psi\rangle$$

$$|\Psi(0)\rangle = |R_i\rangle$$



Task: What part of it will propagate to a final point R_f?

$$\langle R_f|\Psi(t)\rangle = ?$$

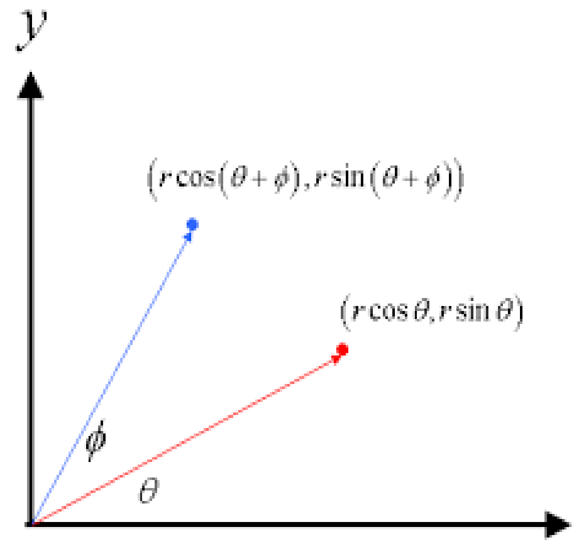
Evolution Operator

$$|\Psi(t)\rangle = \hat{U}(t)|\Psi(0)\rangle$$

Substituting in Schroedinger equation

$$i\hbar\partial_t\hat{U}(t) = \hat{H}\hat{U}(t)$$

$$\hat{U}(0) = \hat{1}$$



If we substitute evolution operator as an exponential

$$\hat{U}(t) = e^{-\frac{i}{\hbar}\hat{H}t}$$

$$|\Psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|R_i\rangle$$

The Matrix Element

$$G(R_i, R_f; t) = \langle R_f | e^{-\frac{i}{\hbar}\hat{H}t} | R_i \rangle$$

The 1-D case

$$G(x_i, x_f; t) = \langle x_f | e^{-\frac{i}{\hbar}\hat{H}t} | x_i \rangle$$

Useful Formulas

$$\hat{1} = \int |x\rangle\langle x| dx$$

$$|\Psi(t)\rangle = \hat{U}(t)|\Psi(0)\rangle = \hat{U}(t-t_1)|\Psi(t_1)\rangle = \hat{U}(t-t_1)\hat{U}(t_1)|\Psi(0)\rangle$$

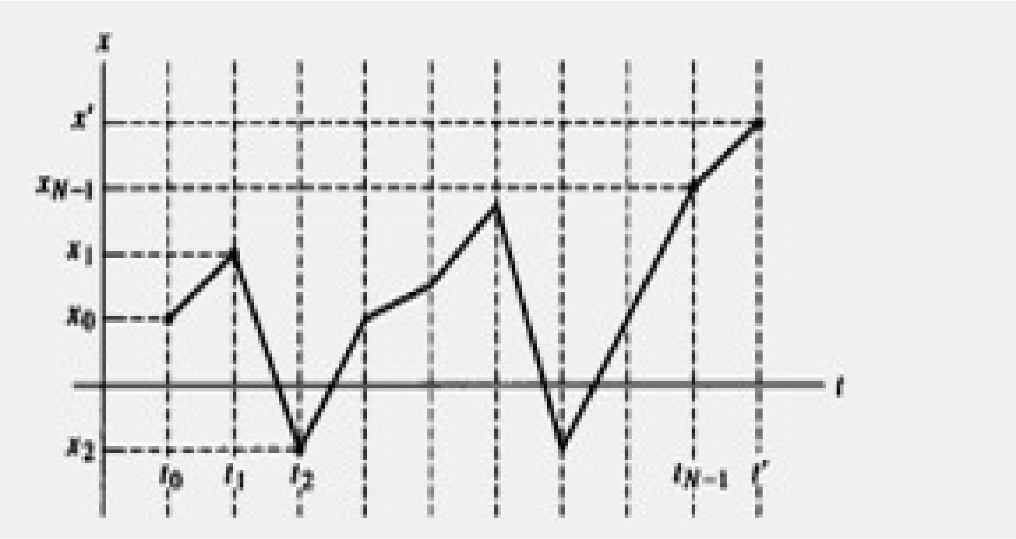
Heading towards the goal...

$$\hat{U}(t) = \hat{U}(t/2)\hat{U}(t/2)$$

$$\langle x_f|\hat{U}(t)|x_i\rangle$$

$$= \int \langle x_f|\hat{U}(t/2)|x\rangle\langle x|\hat{U}(t/2)|x_i\rangle dx$$

Divide in grid



Moving closer...

$$\hat{U}(t) = \hat{U}(t/N) * \hat{U}(t/N) * \cdots Ntimes$$

$$\Delta t = t/N$$

$$e^{-\frac{i}{\hbar}\hat{H}\frac{t}{N}} \approx 1 - \frac{i}{\hbar}\hat{H}\Delta t$$

$$e^{-\frac{i}{\hbar}\hat{H}t} = \left(1 - \frac{i}{\hbar}\hat{H}\frac{t}{N}\right)^N$$

$$\langle x_f|\hat{U}(t)|x_i\rangle = \int dx_1 dx_2 \cdots \prod_{k=0}^{N} \langle x_{k+1}|1 - \frac{i}{\hbar N}\hat{H}t|x_k\rangle$$

Recall QM

$|\Psi\rangle$ vs $\Psi(x)$?

$|\Psi\rangle$: Abstract state of a system (independent of coordinates)

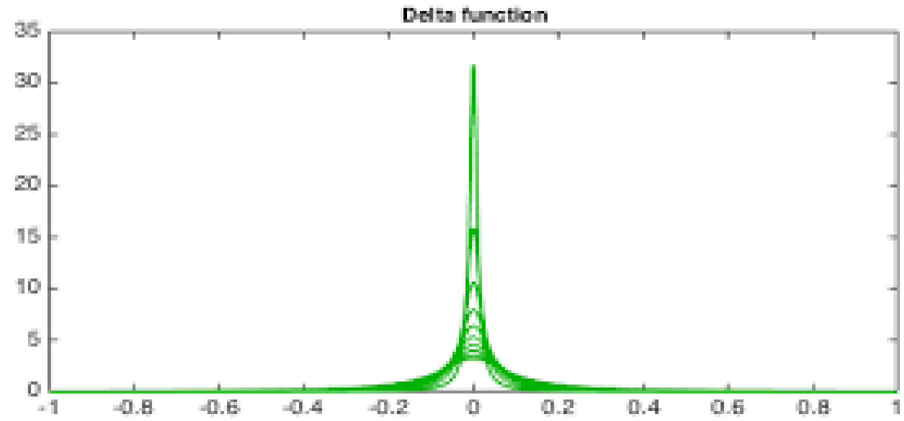
$\Psi(x) = \langle x|\Psi\rangle$: Coordinate dependent representation

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}px}$$

A plane wave (particle) with a given momentum

$\langle x|x\prime\rangle$: Wave function of a particle with a given coordinate

The matrix elements



$$\langle x_{k+1}|x_k\rangle = \delta(x_{k+1} - x_k)$$

$$\langle x_{k+1}|\hat{V}|x_k\rangle = V(x_k)\delta(x_{k+1} - x_k)$$

$$\langle x_{k+1}|\frac{\hat{p}^2}{2m}|x_k\rangle = \int \frac{dp_k}{2\pi\hbar}\frac{{p_k}^2}{2m}e^{\frac{i}{\hbar}p_k(x_{k+1}-x_k)}$$

A bit of math trickery

$$\delta(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} = \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} e^{\frac{i}{\hbar}px}$$

$$\langle x_{k+1} | x_k \rangle = \int \frac{dp_k}{2\pi\hbar} e^{\frac{i}{\hbar}p(x_{k+1}-x_k)}$$

$$\langle x_{k+1} | \hat{V} | x_k \rangle = V(x_k) \int \frac{dp_k}{2\pi\hbar} e^{\frac{i}{\hbar}p(x_{k+1}-x_k)}$$

$$\langle x_{k+1} | \frac{\hat{p}^2}{2m} | x_k \rangle = \int \frac{dp_k}{2\pi\hbar} \frac{{p_k}^2}{2m} e^{\frac{i}{\hbar}p_k(x_{k+1}-x_k)}$$

The final touch: Putting together

$$\langle x_f|\hat{U}(t)|x_i\rangle = \int \cdots \langle x_2|\hat{U}(\Delta t)|x_1\rangle\langle x_1|\hat{U}(\Delta t)|x_i\rangle dx_1 dx_2 \cdots$$

$$\langle x_{k+1}|1 - \frac{i}{\hbar}\hat{H}\Delta t|x_k\rangle \propto \int dp_k e^{\frac{i}{\hbar}p_k \Delta x_k}\left[1 - \frac{i}{\hbar}\left(\frac{{p_k}^2}{2m} + V(x_k)\right)\Delta t\right]$$

$$= e^{\frac{i}{\hbar}\left(\frac{m\Delta x_k^2}{2\Delta t^2} - V(x_k)\right)\Delta t}$$

$$= e^{\frac{i}{\hbar}L_k \Delta t}$$

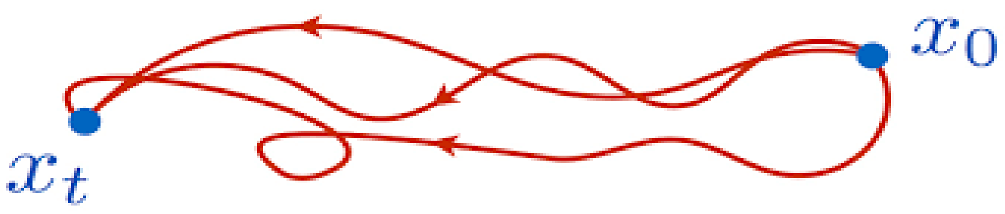
The final touch: The result

$$\langle x_f|\hat{U}(t)|x_i\rangle = \int dx_1 dx_2 \cdots e^{\frac{i}{\hbar}L_1\Delta t} e^{\frac{i}{\hbar}L_2\Delta t} \ldots$$

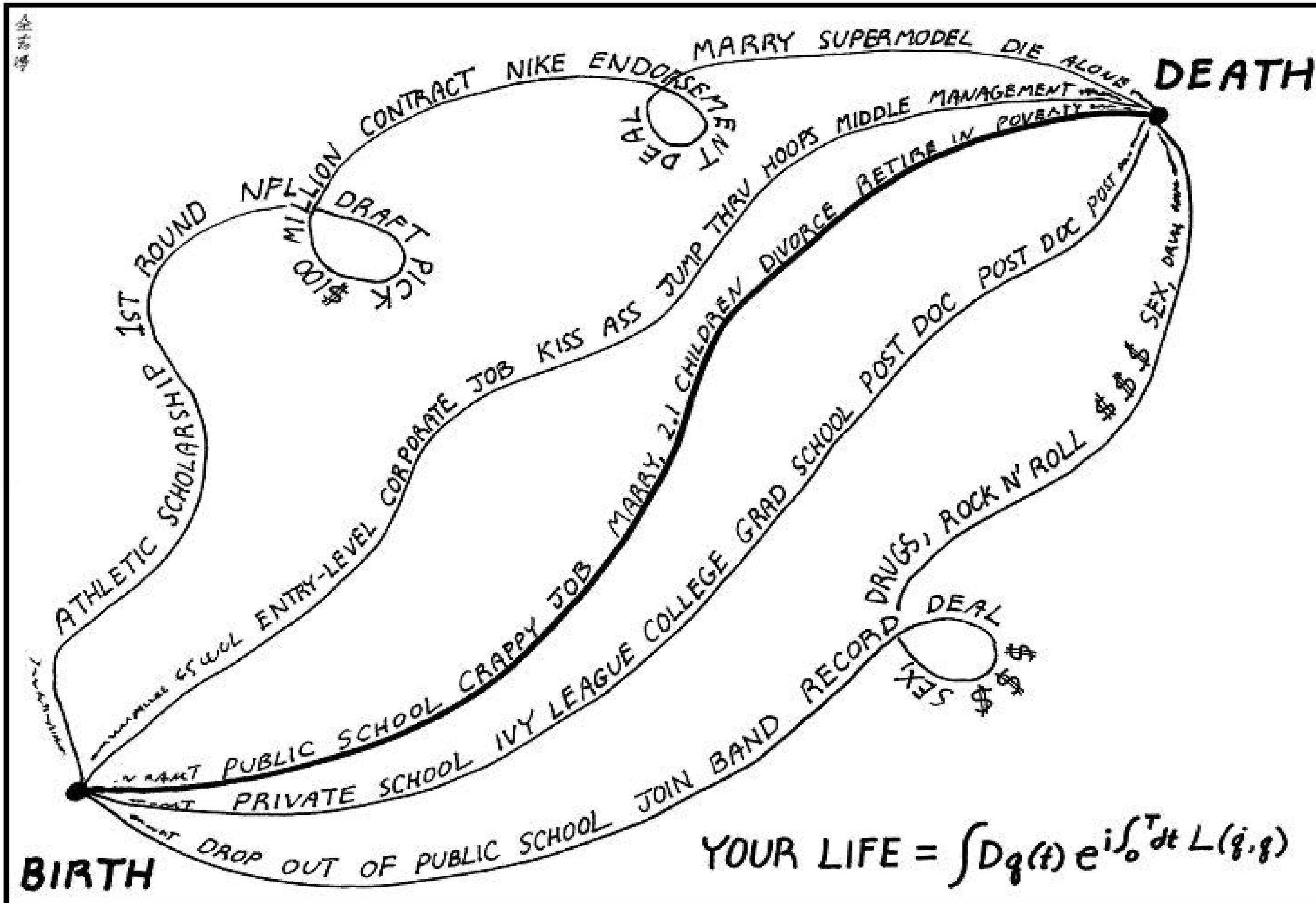
$$e^{\frac{i}{\hbar}\sum_k L_k \Delta t} = e^{\frac{i}{\hbar}\int_0^t L dt} = e^{\frac{i}{\hbar}S}$$

$$\langle x_f|\hat{U}(t)|x_i\rangle = \int Dx(t) e^{\frac{i}{\hbar}\int_0^t L dt}$$

$$\langle x_t|e^{-i\hat{H}t/\hbar}|x_0\rangle =$$



Take home



The Path Integral Formulation of Your Life