Linear Algebra: Practice Problems

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1. Let A and B are similar matrices. Prove:
   a. A^2 and B^2 are also similar.
   b. $(A - \lambda I)$ and $(B - \lambda I)$ are also similar, for any $\lambda \in F$.
2. Suppose that nxn matrices A and B are similar. Then show that the nullity of A is equal to the nullity of B.
3. Let
$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 2 & 3 & 5 \end{bmatrix}$$
   a. Find a matrix B in reduced row echelon form such that B is row equivalent to the matrix A.
   b. Find a basis for the null space of A.
   c. Find a basis for the range of A that consists of columns of A. For each columns, A_j of A that does not appear in the basis, express A_j as a linear combination of the basis vectors.
   d. Find a basis for the row space of A.
4. (a) Find all 3x3 matrices which are in reduced row echelon form and have rank 1.
   (b) Find all such matrices with rank 2.
5. Determine all possibilities for the number of solutions of each of the system of linear equations described below.
   (a) A system of 5 equations in 3 unknowns and it has x1=0, x2=3, x3=1 as a solution.
   (b) A homogeneous system of 5 equations in 4 unknowns and the rank of the system is 4.
6. Let A be a 3x3 matrix. Suppose that A has eigenvalues 2 and 1, and suppose that u and v are eigenvectors corresponding to 2 and 1, respectively, where $u = (1, 0, -1)^T$ and $v = (2, 1, 0)^T$. Compute $A^5 w$ where $w = (7, 2, -3)^T$
7. Prove that all eigenvalues of a real symmetric 2x2 matrix are real.
8. Suppose that A is an nxn matrix with eigenvalue λ and corresponding eigenvector v.

(a) If A is invertible, is v an eigenvector of A^{-1}? If so, what is the corresponding eigenvalue? If not, explain why not.

(b) Is 3v an eigenvector of A? If so, what is the corresponding eigenvalue? If not, explain why not.

9. Let A be an nxn matrix. Suppose that the matrix A^2 has a real eigenvalue λ. Then show that either $\lambda^{0.5}$ or $-\lambda^{0.5}$ is an eigenvalue of the matrix A.