Linear Algebra: Practice Problems

Kushagra Chandak

1. Let A be a matrix with dimensions m x n. Suppose that the nullspace of A is a plane in R^3 and the column space is spanned by a non-zero vector v in R^5. Determine m and n. Also find rank and nullity of A.

2. Let $T : R^n \to R^m$ is a linear transformation. Suppose nullity of T is zero. If $\{x_1, \dots x_k\}$ is a linearly independent subset of R^n then show that $\{T(x_1), \dots T(x_k)\}$ is a linearly independent subset of R^m

3. Let $P_2(R)$ be the vector space ove R consisting of all polynomials with reall coefficients of degree 2 or less. Let $B = \{1, x, x^2\}$ be a basis of this space. Find the matrix representation of a linear transformation T: $P_2(R) \to P_2(R)$ w.r.t to the basis B where T is defined as

$$T(f(x)) = \frac{d^2 f(x)}{dx^2} - 3\frac{df(x)}{dx}$$

4. Let P_n be the vector space of polynomial of degree at most n. The standard basis of P_n is $B = \{1, x, x^2 \dots x^n\}$.
   Let $T : P_3 \to P_5$ be a map defined by

$$T(f)(x) = (x^2 - 2)f(x)$$

   Determine if T is a linear transformation and if it is then find the matrix representation of T relative to standard basis of P_3 and P_5

5. Let $C([-1, 1])$ denote vector space of real valued continuous functions on the interval $[-1, 1]$ Define the subspace

$$W = \{f \in C([-1, 1]) \mid f(0) = 0\}$$

   Define the map $T : C([-1, 1]) \to W$ by $T(f)(x) = f(x) - f(0)$. Determine if T is a linear map and if it is, determine its nullspace and range.

6. Let $W = C^\infty(R)$ be the vector space of all C^∞ real-valued functions. Let V be the vector space of all linear transformations from W to W. Let T_1,T_2,T_3 be the elements in V defined by

$$T_1(f(x)) = \frac{df(x)}{dx}$$

$$T_2(f(x)) = \frac{d^2 f(x)}{dx^2}$$

$$T_3(f(x)) = \int_0^x f(t)dt$$

Determine whether T_1, T_2, T_3 are linearly dependent or independent.

7. Let U and V be vector spaces over a scalar field F. Let T: U→V be a linear transformation. Prove that T is injective (one-to-one) if and only if the nullity of T is zero.

8. Let V denote the vector space of 2 x 2 matrices, and W the vector space of 3 x 2 matrices. Define the linear transformation T: V→ W by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+b & 2d \\ 2b-d & -3c \\ 2b-c & -3a \end{bmatrix}$$

Find a basis for the range of T.