MILP models for RCPSP

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Problem setting and the basic version

- Combinatorial optimization problem to schedule activities on resources that are limited in quantity. Defined by the 6-tuple (V, p, E, R, B, b).
- S_i = Start time of i-th activity, $i = 0 \dots n+1$. S_{n+1} is date of the project completion time, called makespan.
- Precedence constraints: $S_j - S_i \geq p_i \quad \forall (i,j) \in E$
- Resource constraints: $\sum_{i \in A_t} b_{ik} \leq B_k \quad \forall k \in R, \forall t \in H$
- Feasible schedule S (with i-th component S_i): Compatible with both the above constraints.
- Objective: Find a schedule S of minimal makespan subject to precedence and resource constraints.
- RCPSP is NP-hard in strong sense.

RCPSP with consumption and production of resources: RCPSP/CPR

- Can also have cumulative resources: can be consumed at the start of an activity in certain amount (c_{ip}^{-}) and/or then produced in another amount at the end (c_{ip}^{+}).
- ES_i and LS_i can be calculated in polynomial time during preprocessing. So $[ES_i, LS_i]$ represents the time window during which activity i can start.

Formulations for RCPSP

- Time-indexed: $x_{it} = 1$ iff activity i starts starts at time t, otherwise 0.
  - Discrete time (DT) and disaggregated discrete time (DDT).
- Flow-based continous-time: $A_0 =$ resource source and $A_{n+1} =$ resource sink. f_{ijk}: quantity of resource k transferred from activity i to j.
- On/off event based: An event occurs when an activity starts or ends. $z_{ie} = 1$ iff activity i starts at e or is in process at e.

Discrete time (DT) formulation for RCPSP

$$\min \sum_{t=ES_{n+1}}^{LS_{n+1}} t x_{n+1,t}$$

subject to

$$\sum_{t=ES_j}^{LS_j} t x_{jt} \geq \sum_{t=ES_i}^{LS_i} t x_{it} + p_i \qquad \forall (i,j) \in E$$

$$\sum_{i=1}^{n} b_{ik} \sum_{\tau=\max(ES_i, t-p_i+1)}^{n} x_{i\tau} \leq B_k \qquad \forall t \in H, \forall k \in R$$

$$\sum_{t=ES_i}^{LS_i} x_{it} = 1 \qquad \forall i \in A \cup \{n+1\}$$

$$\begin{aligned}
&x_{00} = 1 \\
&x_{it} = 0 && \forall i \in A \cup \{n+1\}, t \in H \setminus \{ES_i, LS_i\} \\
&x_{it} \in \{0,1\} && \forall i \in A \cup \{n+1\}, \forall t \in \{ES_i, LS_i\}
\end{aligned}$$

Extension of DT to RCPSP/CPR and DDT

$$
\begin{array}{ll}
s_{0p} = C_p - \sum_{i=1}^{n} x_{i0} c_{ip}^{-} & \forall p \in P \\
s_{tp} = s_{t-1,p} - \sum_{i=1}^{n} x_{it} c_{ip}^{-} + \sum_{i=1}^{n} x_{i,t-p_i} c_{ip}^{+} & \forall (t,p) \in H \times P, t > 0 \\
s_{tp} \geq 0 & \forall (t,p) \in H \times P
\end{array}
$$

Disaggregated Discrete Time (DDT) Formulation:

$$
\sum_{\tau=t}^{LS_i} x_{i\tau} + \sum_{\tau=ES_j}^{\min\left(LS_j, t+p_i-1\right)} x_{j\tau} \leq 1, \quad \forall (i,j) \in E, \forall t \in \{ES_i, LS_i\}
$$

FCT

- Uses flow variables to manage resources: f_{ijk}: quantity of resource k transferred from activity i to activity j.
- Sequantial binary variables: $x_{ij} = 1$ if activity i is processed before j.
- Continous start time variables, S_i: Start time of activity i.
- Since A_0 is source and A_{n+1} is sink, define $\widetilde{b}_{ik} = b_{ik}$ and $\widetilde{b}_{0k} = \widetilde{b}_{n+1,k} = B_k$

Flow based continuous time (FCT) formulation

$$\min S_{n+1}$$

$$\begin{array}{ll} x_{ij} + x_{ji} \leq 1, & \forall (i,j) \in (A \cup \{0, n+1\})^2, i < j \\ x_{ik} \geq x_{ij} + x_{jk} - 1 & \forall (i,j,k) \in (A \cup \{0, n+1\})^3 \end{array}$$

$$S_j - S_i \geq -M_{ij} + (p_i + M_{ij})\, x_{ij} \quad \forall (i,j) \in (A \cup \{0, n+1\})^2$$

$$f_{ijk} \leq \min\left(\tilde{b}_{ik}, \tilde{b}_{jk}\right) x_{ij} \quad \forall (i,j) \in (A \cup \{0\} \times A \cup \{n+1\}), \forall k \in R$$

$$\sum_{j \in A \cup \{0, n+1\}} f_{ijk} = \tilde{b}_{ik} \quad \forall i \in A \cup \{0, n+1\}, \forall k \in R$$

$$\sum_{i \in A \cup \{0, n+1\}} f_{ijk} = \tilde{b}_{jk} \quad \forall i \in A \cup \{0, n+1\}, \forall k \in R$$

$$f_{n+1,0,k} = B_k$$

$$x_{ij} = 1$$
$$x_{ji} = 0$$

$$f_{ijk} \geq 0 \qquad \forall (i, j) \in (A \cup \{0, n+1\})^2, \forall k \in R$$

$$S_0 = 0$$
$$ES_i \leq S_i \leq LS_i \quad \forall i \in A \cup \{n+1\}^2$$
$$x_{ij} \in \{0, 1\} \qquad \forall (i, j) \in (A \cup \{0, n+1\})^2$$

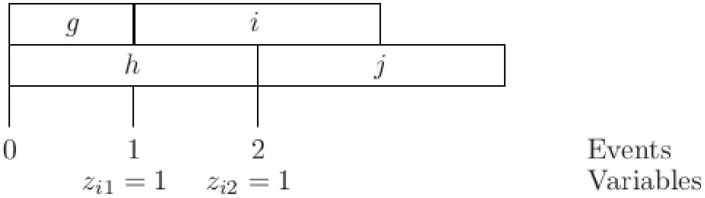
Extension of FCT to RCPSP/CPR

$$D_{ijp} \leq \min\left(c_{ip}^{+}, c_{jp}^{-}\right) x_{i,j} \quad \forall (i,j) \in A^2, p \in P$$

$$\begin{array}{ll} \sum_{i \in A \cup \{0\}} D_{ijp} = c_{jp}^{-} & \forall j \in A \cup \{n+1\}, p \in P \\ \sum_{j \in A \cup \{n+1\}} D_{ijp} = c_{ip}^{+} & \forall i \in A \cup \{0\}, p \in P \end{array}$$

OOE

- RCPSP/CPR formulation with variables indexed by *events* which occur when an activity starts or ends.
- Decision variables z_{ie}, continous variables t_e and C_{max}
- For non-renewable resources: s_{ep} = quantity of resource p at event e, p_{iep} = quantity of p *produced* by activity i at event e, u_{iep} = quantity of p *consumed* by i at e.



On/off event based formulation (OOE)

$$\min C_{\max}$$

subject to

$$\sum_{e \in \mathcal{E}} z_{ie} \geq 1$$

$$C_{\max} \geq t_e + \left(z_{ie} - z_{i(e-1)}\right) p_i \qquad \forall e \in \mathcal{E}, \forall i \in A$$

$$t_0 = 0$$

$$t_{e+1} \geq t_e$$

$$t_f \geq t_e + ((z_{ie} - z_{i,e-1}) - (z_{if} - z_{i,f-1}) - 1)\, p_i \quad \forall (e,f,i) \in \mathcal{E}^2 \times A, f > e$$

$$\sum_{e'=0}^{e-1} z_{ie'} \leq e\left(1 - \left(z_{ie} - z_{i(e-1)}\right)\right) \qquad \forall i \neq A, \quad \forall e \neq 0 \in \mathcal{E}$$
$$\sum_{e'=0}^{n-1} z_{ie'} \leq (n-e)\left(1 + \left(z_{ie} - z_{i(e-1)}\right)\right) \quad \forall i \neq A, \quad \forall e \neq 0 \in \mathcal{E}$$

$$z_{ie} + \sum_{e'=0}^{e} z_{je'} \leq 1 + (1 - z_{ie})\, e \qquad \forall e \in \mathcal{E}, \forall (i,j) \in E$$

$$\sum_{i=0}^{n-1} b_{ik} z_{ie} \leq B_k$$

$$\begin{aligned} p_{iep} &\geq 0 && \forall (e,i,p) \in \mathcal{E} \times A \times P \\ p_{iep} &\geq c^{+}_{ip} (z_{i,e-1} - z_{i,e}) && \forall (e,i,p) \in \mathcal{E} \times A \times P \end{aligned}$$

$$p_{iep} \leq c^{+}_{ip} z_{i,e-1} \qquad \forall (e,i,p) \in \mathcal{E} \times A \times P$$

$$p_{iep} \leq c^{+}_{ip} (1 - z_{i,e}) \quad \forall (e,i,p) \in \mathcal{E} \times A \times P$$

$$u_{iep} \geq 0 \qquad \forall (e,i,p) \in \mathcal{E} \times A \times P$$

$$u_{iep} \geq c^{-}_{ip} (z_{i,e} - z_{i,e-1}) \quad \forall (e,i,p) \in \mathcal{E} \times A \times P$$

$$u_{iep} \leq c^{-}_{ip} z_{ie} \qquad \forall (e,i,p) \in \mathcal{E} \times A \times P$$

$$u_{iep} \leq c^{-}_{ip} \left(1 - z_{i,e-1}\right) \qquad \forall (e, i, p) \in \mathcal{E} \times A \times P$$

$$s_{ep} = s_{e-1,p} + \sum_{i \in A} p_{iep} - \sum_{i \in A} u_{iep} \quad \forall (e, p) \in \mathcal{E} \times P, e > 0$$

$$s_{0p} = C_p - \sum_{i \in A} u_{i0p}$$

OOE_Prec

- From the set of possible events for an activity, remove all the first events during which an activity cannot be in process because of its predecessors. Similarly, for successors.
- $z_{ie} = 0, \quad i \in A, e \in \{0, \ldots, |A(i)|\} \cup \{n - |D(i)| + 1, \ldots, n\}$
- Eliminating decision variables before setting up the OOE formulation gives us OOE_Prec formulation.